ON THE SIMPLICITY OF LIE ALGEBRA OF LEAVITT PATH ALGEBRA

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Abstract. For a field F and a row-finite directed graph Γ let $L(\Gamma)$ be the Leavitt path algebra. We find necessary and sufficient conditions for the Lie algebra $[L(\Gamma), L(\Gamma)]$ to be simple.

1. Introduction.

In [3] G. Abrams and Z. Mesyan found necessary and sufficient conditions for a simple Leavitt path algebra $L(\Gamma)$ to give rise to a simple Lie algebra $[L(\Gamma), L(\Gamma)]$. This result is based on a simple easily checkable criterion for a linear combination of vertices $\sum_i \alpha_i v_i, \alpha_i \in F, v_i \in V$, to lie in $[L(\Gamma), L(\Gamma)]$. In this paper we extend the result of G. Abrams and Z. Mesyan to not necessarily simple algebras and find the necessary and sufficient conditions for a Lie algebra $[L(\Gamma), L(\Gamma)]$ to be simple.

2. Definitions and Terminology

A (directed) graph $\Gamma = (V, E, s, r)$ consists of two sets V and E that are respectively called vertices and edges, and two maps s, $r : E \to V$.The vertices $s(e)$ and $r(e)$ are referred to as the source and the range of the edge e, respectively. The graph is called row-finite if for all vertices $v \in V$,$card(s^{-1}(v)) < \infty$. A vertex v for which $s^{-1}(v) = \emptyset$ is called a sink. A vertex v such that $r^{-1}(v) = \emptyset$ is called a source. A path $p = e_1.....e_n$ in a graph Γ is a sequence of edges $e_1.....e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, ..., n-1$. In this case we say that the path p starts at the vertex $s(e_1)$ and ends at the vertex $r(e_n)$. If $s(e_1) = r(e_n)$, then the path is closed. If $p = e_1.....e_n$ is a closed path and the vertices $s(e_1)$, ..., $s(e_n)$ are distinct, then the subgraph ($s(e_1), ..., s(e_n); e_1, ..., e_n$) of the graph Γ is called a cycle. A cycle of length 1 is called a loop.

Definition 1. Let W be a subset of V. We say that

- W is hereditary if $v \in W$ implies $w \in W$ for every vertex w connects to v.
- W is saturated if $\{r(e) : s(e) = v\} \subseteq W$ implies that $v \in W$, for every non-sink vertex $v \in V$.

Definition 2. We call an edge $e \in E$ a *fiber* if $s(e)$ is source, $r(e)$ is sink and $E(V, r(e)) = \{e\}$.

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Definition 3. We call a vertex v in a connected graph $\Gamma(V,E)$ a *balloon* over a nonempty subset W of V if (i) $v \notin W$, (ii) there is a loop $C \in E(v,v)$, (iii) $E(v,W) \neq \emptyset$, (iv) $E(v,V) = \{C\} \cup E(v,W)$, and (v) $E(V,v) = \{C\}$.

Let Γ be a row-finite graph and let F be a field. The Leavitt path F-algebra $L(\Gamma)$ is the F-algebra presented by the set of generators $\{v | v \in V\}$, $\{e, e^* |\ e \in E\}$ and the set of relators (1) $v_i v_j = \delta_{v_i,v_j} v_i$ for all $v_i, v_j \in V$; (2) $s(e)e = er(e) = e$, $r(e)e^* = e^* s(e) = e^*$ for all $e \in E$; (3) $e^* f = \delta_{e,f} r(e)$, for all $e,\ f \in E$; (4) $v = \sum_{s(e)=v} ee^*$, for an arbitrary vertex v which is not a sink. The mapping which sends v to v for $v \in V$, e to e^* and e^* to e for $e \in E$, extends to an involution of the algebra $L(\Gamma)$. If $p = e_1.....e_n$ is a path, then $p^* = e_n^*....e_1^*$. In what follows we consider only row-finite directed graphs. We call a graph Γ simple if the Leavitt path algebra $L(\Gamma)$ is simple. The conditions for a graph to be simple are given in [1].

Let A be an associative $F-$algebra. For elements $a, b \in A$, let $[a,b] = ab - ba$ be their the commutator. Then $A^{(-)} = (A, [,])$ is a Lie algebra. If A is an associative algebra and S is a subset of A, we will denote the ideal of A generated by S as $id_A(S)$.

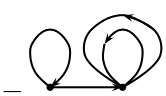
3. Lie algebra of Leavitt path algebra

We start with theorem by G. Abrams and Z. Mesyan in [3].

Theorem 1. *([3]) Let $\Gamma(V,E)$ be a directed graph. Let $L(\Gamma)$ be a simple algebra.*

(i) *If V is infinite then the Lie algebra $[L(\Gamma), L(\Gamma)]$ is simple;*
(ii) *If V is finite, then $[L(\Gamma), L(\Gamma)]$ is simple if and only if $1_{L(\Gamma)} = \sum_{v \in V} v \notin [L(\Gamma), L(\Gamma)]$.*

There exist however non-simple Leavitt path algebras having the Lie algebra $[L(\Gamma), L(\Gamma)]$ simple.

Example 1. Let $\Gamma =$  . The Lie algebra $[L(\Gamma), L(\Gamma)]$ is isomorphic to the Lie algebra of infinite finitary matrices over the Leavitt algebra $L(2)$ and therefore is simple.

The following theorem gives a classification of directed graph having $[L(\Gamma), L(\Gamma)]$ simple.

Theorem 2. *Let $\Gamma(V,E)$ be a directed row-finite graph. The Lie algebra $[L(\Gamma), L(\Gamma)]$ is simple if and only if either $L(\Gamma)$ is simple- this case is covered by Theorem1 - or Γ contains a simple subgraph W such that every point $v \in V \setminus W$ is a balloon over W, and* $\sum_{w \in r(E(v,W))} w \in [L(W), L(W)]$.

We will prove the theorem by proving a series of lemmas. The first lemma is due to G. Abrams and Z. Mesyan, [3]. We will state it without proof.

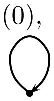
Lemma 1. *([3]) Let $\Gamma(V,E)$ be a directed graph. Then $[L(\Gamma),L(\Gamma)]=(0)$ if and only if Γ is a disjoint union of* $\bullet$,  *(vertices and loops).*

Lemma 2. *Let $\Gamma(V,E)$ be a row-finite graph. If the Lie algebra $[L(\Gamma),L(\Gamma)]$ is nonzero simple, then every cycle has an exit.*

Proof. Let C be a no exist cycle of Γ of length d. Then $L(C)\cong M_d(F[t,t^{-1}])$. Let a be the sum of all vertices on the cycle C. The element a is the identity of $L(C)$ and $L(C)=aL(\Gamma)a$. Consider the ideal $J_n=(1-t)^nF[t,t^{-1}]$ of $F[t,t^{-1}]$. Now, if $d\geq 2$, then $[M_d(J_n),M_d(J_n)]\neq(0)$ for all $n\geq 1$, see [5]. Let $I_n=id_{L(\Gamma)}(M_d(J_n))$. Then $[I_n,I_n]\lhd[L(\Gamma),L(\Gamma)]$ and because of simplicity of $[L(\Gamma),L(\Gamma)]$ we have $[L(\Gamma),L(\Gamma)]=[I_n,I_n]\subseteq I_n$. Hence $[L(\Gamma),L(\Gamma)]\cap L(C)\subseteq I_n\cap L(C)=M_d(J_n)$. Since $\cap_nJ_n=(0)$, it follows that $[L(\Gamma),L(\Gamma)]\cap L(C)=(0)$, but $(0)\neq[M_d(F[t,t^{-1}]),M_d(F[t,t^{-1}])]\subseteq[L(\Gamma),L(\Gamma)]\cap L(C)$. A contradiction. Hence $d=1$. Thus C is a loop. Since C has no exit and can not be isolated there exist an edge $e\in E$, such that $s(e)\notin V(C)=\{v\}$. Let $J_n=(v-C)^nL(C)$, $I_n=id_{L(\Gamma)}(J_n)$, $vI_nv\subseteq J_n$. Now, $[eJ_n,J_n]=eJ_n\neq(0)$. Hence $[I_n,I_n]\neq(0)$, $[L(\Gamma),L(\Gamma)]=[I_n,I_n]\subseteq I_n$ and therefore $v[L(\Gamma),L(\Gamma)]v\subseteq J_n$. Since $\cap_nJ_n=(0)$ it follows that $v[L(\Gamma),L(\Gamma)]v=(0)$, but $[e^*,e]=v-ee^*$, and $v[e^*,e]v=v\neq 0$. A contradiction. □

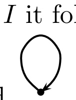
The algebra $L(\Gamma)$ is graded: $dg(v)=0$, $dg(e)=1$, $dg(e^*)=-1$ for all $v\in V,e\in E$. In [9] it is shown that every graded ideal I of $L(\Gamma)$ is generated (as an ideal) by $I\cap V$. Thus there is a one-to-one correspondence between graded ideals and hereditary saturated subsets of V.

Lemma 3. *Let $\Gamma(V,E)$ be a row-finite graph. Let W be nonempty hereditary and saturated subset of V. Let $I=id_{L(\Gamma)}(W)$. If $[L(\Gamma),L(\Gamma)]$ is nonzero simple, then $[I,I]\neq(0)$.*

Proof. If $[I,I]=(0)$, then, in particular, $[L(W),L(W)]=(0)$, W is a disjoint union of $\bullet$, and  . This implies that for every vertex $w\in W$ there exist an edge $e\in E$ such that $r(e)=w$, $s(e)\notin W$ otherwise w is isolated in Γ. Now, $e,e^*\in I$ and $[e,e^*]\neq 0$. Lemma is proved. □

Lemma 4. *Let $\Gamma(V,E)$ be a a row-finite graph. If $[L(\Gamma),L(\Gamma)]$ is nonzero simple, then there exists a minimal hereditary saturated subset in V.*

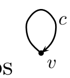
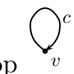
Proof. We need to show that the intersection of all nonzero graded ideals in $L(\Gamma)$ is nonzero. If I is a nonzero graded ideal of $L(\Gamma)$ then by Lemma 3 $[I,I]\neq(0)$. Since $[L(\Gamma),L(\Gamma)]$ is simple, then $[L(\Gamma),L(\Gamma)]=[I,I]$ and therefore $[L(\Gamma),L(\Gamma)]$ lies in the intersection of all nonzero graded ideals of $L(\Gamma)$. □

Let $\Gamma(V,E)$ be a a row-finite graph. Suppose $[L(\Gamma),L(\Gamma)]$ is nonzero simple. Let W be a minimal hereditary saturated subset in V. Let $I=id_{L(\Gamma)}(W)$, $\Gamma'=(V\setminus W,E\setminus E(V,W))$. We assume that $W\neq V$, that is $L(\Gamma)$ is not simple. Since $L(\Gamma')\cong L(\Gamma)/I$ and $[L(\Gamma),L(\Gamma)]\subseteq I$ it follows that $[L(\Gamma'),L(\Gamma')]=(0)$. By Lemma 1 Γ' is a disjoint union of $\bullet$, and  .

Lemma 5. *Γ' does not have components* $\bullet$.

Proof. Let a vertex $v \in V \setminus W$ be isolated in Γ'. Then $E(V \setminus W, v) = \emptyset$. Since W is hereditary and $v \notin W$ we conclude that $E(V, v) = \emptyset$. Since v can not be isolated in Γ it can not be a sink, $E(v, V) \neq \emptyset$. But $E(v, V \setminus W) = \emptyset$, hence all descendants of v lie in W. Since W is saturated we conclude that $v \in W$, a contradiction. $\square$

Lemma 6. *Every vertex $v \in V \setminus W$ is a balloon over W.*

Proof. By what we have shown Γ' is a disjoint union of loops  . It is easily to see that $E(V, v) = \{c\}$ and $E(v, V \setminus W) = \{c\}$. If $E(v, W) = \emptyset$ then the loop  is isolated in Γ. Hence $E(v, W) \neq \emptyset$. Thus v is a balloon over W. $\square$

Let S_0 be the span of all elements pp^*, where p is a path on Γ including pathes of length zero(that is vertices). Let S_1 be the span of all elements pq^*, where p, q are pathes on Γ, $r(p) = r(q)$, $p \neq q$. It follows from the description of a Groebner - Shirshov basis of $L(\Gamma)$ [4] that $L(\Gamma) = S_0 + S_1$ is a direct sum of vector spaces. Let M be the semigroup generated by $V \cup E \cup E^*$. It is easily to see that (i) $M = (M \cap S_0) \cup (M \cap S_1)$, (ii) for arbitrary elements $a, b \in M$ if $0 \neq ab \in S_i$, then $ba \in S_i$ or $ba = 0$, for $i = 0, 1$.

Lemma 7. $[I, I] \cap S_0 = span\{[p, p^*] |$ *p is a path on Γ , $r(p) \in W$*$\}$.

Proof. The ideal I is spanned by elements pq^*; p, q are paths, $r(p) = r(q) \in W$. Consider two such elements $p_1q_1^*$ and $p_2q_2^*$, $0 \neq p_1q_1^*p_2q_2^* \in S_0$. Since $q_1^*p_2 \neq 0$ it follows that $p_2 = q_1u$ or $q_1 = p_2u$, where u is a path on Γ. Consider the first case, $p_2 = q_1u$. Then $p_1q_1^*p_2q_2^* = p_1uq_2^*$. Since this element lies in S_0 we conclude that $q_2 = p_1u$. Now, $p_2q_2^*p_1q_1^* = q_1uu^*p_1^*p_1q_1^* = (q_1u)(q_1u)^*$ and therefore $[p_1q_1^*, p_2q_2^*] = (p_1u)(p_1u)^* - (q_1u)(q_1u)^* = [p_1u, (p_1u)^*] - [q_1u, (q_1u)^*]$. Remember that $r(u) = r(q_2) \in W$. Let $q_1 = p_2u$. Then $p_1q_1^*p_2q_2^* = p_1u^*p_2^*p_2q_2^* = p_1(q_2u)^*$. Again $p_1q_1^*p_2q_2^* \in S_0$ implies $p_1 = q_2u$. Now, $p_2q_2^*p_1q_1^* = p_2q_2^*q_2uu^*p_2^* = (p_2u)(p_2u)^*$. Therefore, $[p_1q_1^*, p_2q_2^*] = (q_2u)(q_2u)^* - (p_2u)(p_2u)^* = [q_2u, (q_2u)^*] - [p_2u, (p_2u)^*]$ and $r(u) = r(p_1) \in W$. $\square$

Let $v \in V \setminus W$, $E(v, W) = \{e_1, \ldots, e_n\}$, $r(e_i) = w_i$ for $1 \leq i \leq n$. Let $w = \sum_{i=1}^{n} w_i$.

Lemma 8. $w \in [L(W), L(W)]$.

Proof. Since v is a balloon over W, let c be the loop from $E(v, v)$, we have $v = cc^* + \sum_{i=1}^{n} e_ie_i^*$. Hence $c^*c - cc^* = v - (v - \sum_{i=1}^{n} e_ie_i^*) = \sum_{i=1}^{n} e_ie_i^* = \sum_{i=1}^{n} [e_i, e_i^*] + \sum_{i=1}^{n} e_i^*e_i = \sum_{i=1}^{n} [e_i, e_i^*] + w$. Thus $w = c^*c - cc^* - \sum_{i=1}^{n} [e_i, e_i^*] = [c^*, c] - \sum_{i=1}^{n} [e_i, e_i^*] \in [L(\Gamma), L(\Gamma)] = [I, I]$. Hence $w \in [I, I] \cap S_0$. By Lemma 7 $w = \sum_i \alpha_i [p_i, p_i^*]$, $\alpha_i \in F$, $r(p_i) \in W$.

We will distinguish between pathes that start with an edge from $E(V \setminus W, W)$ and paths that lie entirely on W, $w = \sum_i \alpha_{e,i} [ep_{e,i}, p_{e,i}^* e^*] + \sum \beta [q, q^*]$, where e runs over

$E(V \backslash W, W)$, $\alpha_{e,i} \in F$, $p_{e,i}$ and q are paths on W. We have, $w = \sum_i \alpha_{e,i}(ep_{e,i}p_{e,i}^* e^* - r(p_{e,i})) + \sum \beta[q, q^*]$. Fix $e \in E(V \setminus W, W)$. From the description of the basis of $L(\Gamma)$ in [4] it follows that $\sum_i \alpha_{e,i} ep_{e,i}p_{e,i}^* e^* = 0$ and therefore $\sum_i \alpha_{e,i} p_{e,i}p_{e,i}^* = 0$. Now $\sum_i \alpha_{e,i}(ep_{e,i}p_{e,i}^* e^* - r(p_{e,i})) = \sum_i \alpha_{e,i}[p_{e,i}, p_{e,i}^*] \in [L(W), L(W)]$.
Hence $w = \sum \alpha_{e,i}[p_{e,i}, p_{e,i}^*] + \sum \beta[q, q^*] \in [L(W), L(W)]$. □

We proved Theorem 2 in one direction.

4. Simplicity of the Lie algebra of Leavitt path algebra

Let $\Gamma(V, E)$ be a graph. Suppose that $W \subsetneq V$ is a simple subgraph, every vertex $v \in V \setminus W$ is a balloon over W and $\sum_{w \in r(E(v,W))} w$ lies in $[L(W), L(W)]$. We will show that the algebra $[L(\Gamma), L(\Gamma)]$ is simple. As above, denote $I = id_{L(\Gamma)}(W)$. The following lemma was proved in [5].

Lemma 9. *I is a simple algebra.*

Lemma 10. *Let A be an arbitrary simple algebra with two orthogonal idempotents e_1, e_2. Then $A = [A, A] + e_i A e_i$, $i = 1, 2$.*

Proof. We have $A = Ae_1A$. For arbitrary elements $a, b \in A$, $ae_1b = [a, e_1b] + e_1ba$. Similarly, $A = Ae_2A$. For arbitrary elements $a, b \in A$, we have $e_1ae_2b = [e_1ae_2, e_2b] + e_2be_1ae_2$. We proved that $A = [A, A] + e_2Ae_2$. The equality $A = [A, A] + e_1Ae_1$ is proved similarly. □

Lemma 11. $[L(\Gamma), L(\Gamma)] = [I, I]$.

Proof. We have $L(\Gamma) = I + span\{c_v^n | n \geq 0, v \in V \setminus W\} + span\{(c_v^*)^n | n \geq 1, v \in V \setminus W\}$. Let $w \in W$. Then, by Lemma 10, $I = [I, I] + wIw$.
Hence $[c_v^n, I] = [c_v^n, [I, I] + wIw] = [c_v^n, [I, I]] \subseteq [I, I]$. Similarly, $[(c_v^*)^n, I] \subseteq [I, I]$. It remains to show that $[c_v^n, (c_v^*)^m] \in [I, I]$. Let $c = c_v$. Suppose at first that $m > n$. Then

$$
\begin{aligned}
[c^n, (c^*)^m] &= c^n(c^*)^m - (c^*)^{m-n} \\
&= c^{n-1}(cc^*)(c^*)^{m-1} - (c^*)^{m-n} \\
&= c^{n-1}(v - \sum e_i e_i^*)(c^*)^{m-1} - (c^*)^{m-n} \\
&= (c^{n-1}(c^*)^{m-1} - (c^*)^{m-n}) - c^{n-1} \sum e_i e_i^* (c^*)^{m-1}.
\end{aligned}
$$

The first summand $c^{n-1}(c^*)^{m-1} - (c^*)^{m-n} = [c^{n-1}, (c^*)^{m-1}]$ and we can apply the induction assumption. Furthermore,
$c^{n-1}e_ie_i^*(c^*)^{m-1} = [c^{n-1}e_i, e_i^*(c^*)^{m-1}] + e_i^*(c^*)^{m-1}c^{n-1}e_i = [c^{n-1}e_i, e_i^*(c^*)^{m-1}]$, since $e_i^*(c^*)^{m-1}c^{n-1}e_i = e_i^*(c^*)^{m-n}e_i = 0$. Now, let $n > m$. Then

$$
\begin{aligned}
[c^n, (c^*)^m] &= c^n(c^*)^m - c^{n-m} \\
&= c^{n-1}(v - \sum e_i e_i^*)(c^*)^{m-1} - c^{n-m} \\
&= [c^{n-1}, (c^*)^{m-1}] - c^{n-1} \sum e_i e_i^* (c^*)^{m-1}.
\end{aligned}
$$

As above,

$$\begin{aligned} c^{n-1}e_ie_i^*(c^*)^{m-1} &= [c^{n-1}e_i, e_i^*(c^*)^{m-1}] + e_i^*(c^*)^{m-1}c^{n-1}e_i \\ &= [c^{n-1}e_i, e_i^*(c^*)^{m-1}] + e_i^*c^{n-m}e_i = [c^{n-1}e_i, e_i^*(c^*)^{m-1}]. \end{aligned}$$

Finally, let $n = m$. As above we conclude that

$$[c^n, (c^*)^n] = [c^{n-1}, (c^*)^{n-1}] - c^{n-1}\sum e_ie_i^*(c^*)^{n-1},$$

$\sum c^{n-1}e_ie_i^*(c^*)^{n-1} = \sum[c^{n-1}e_i, e_i^*(c^*)^{n-1}] + \sum e_i^*e_i \in [I, I]$ by our assumption. Lemma is proved. □

Lemma 12. *The algebra $[I, I]$ has zero center.*

Proof. I. Herstein [7] proved that in a simple associative algebra A of dimension bigger than 4 over its center, $[A, A]$ generates A. Hence an elements from I, that commutes with $[I, I]$, lies in the center of I. An arbitrary element from I looks as $z = a_0 + \sum\limits_{e\in E(v_i,W)} ea_e + \sum\limits_{e\in E(v_i,W)} b_e^*e^* + \sum\limits_{\substack{e\in E(v_i,W)\\ f\in E(v_j,W)}} ea_{e,f}f^*$, $a_0, a_e, b_e, a_{e,f} \in L(W)$. Suppose that z lies in the center of I. Commuting z with idempotents $w \in W$, $ee^*, e \in E(v_i, W)$ we see that $z = a_0 + \sum\limits_{e\in E(v_i,W)} ea_ee^*$. This implies that a_0 lies in the center of W. Therefore by [2], $|W| < \infty$ and $a_0 = \alpha \sum\limits_{w\in W} w$, $\alpha \in F$. Multiplying z on the left by e^* and on the right by e, $e \in E(v_i, W)$, we get $r(e)z = a_e = \alpha r(e)$. We proved that $z = \alpha(\sum\limits_{w\in W} w + \sum\limits_{e\in E(v_i,W)} ee^*)$. Now choose a vertex $v_i \in V \setminus W$ and an edge $f \in E(v_i, W)$. We have $zc_if = 0$, whereas $c_ifz = \alpha c_if$. Hence $\alpha = 0$. Lemma is proved.

□

Now it remans to refer to Herstein's theorem about simplicity of $[I, I]/center$, see [8]. Hence $[I, I]$ is simple and therefore $[L(\Gamma), L(\Gamma)]$ is simple.

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